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The Influence of Pitch and Twist on Blade Vibrations

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Nomenclature

$(EI)_{\bar v}$ = principal edgewise flexural rigidity, lb-ft^2
$(EI)_{\bar w}$ = principal flatwise flexural rigidity, lb-ft^2
R = span of blade, ft
T = kinetic energy, ft-lb
U_f = potential energy in centrifugal force field, ft-lb
U_S = strain energy in bending, ft-lb
V = in-plane modal deflection function, ft
W = out-of-plane modal deflection function, ft
m = mass per unit length slug/ft
$m_{w,v}$ = generalized mass, slug
k = generalized spring rate, lb/ft
q = generalized coordinate
r = spanwise position along blade, ft
t = time, sec
v = deflection of blade in plane of rotation, ft
$\bar v$ = deflection of blade in edgewise principal direction, ft
w = deflection of blade normal to plane of rotation, ft
$\bar w$ = deflection of blade in flatwise principal direction, ft
Δ = frequency perturbation parameter, rad/sec
σ = first mass moment of blade, slug-ft
θ = inclination of principal plane of bending, blade geometric pitch angle, rad
η = dummy variable of integration, ft
ω = frequency of vibration, rad/sec
Ω = blade steady frequency of rotation, rad/sec
$\frac{d(\ )}{dt}$ = derivative of ( ) with respect to time
$\frac{\partial(\ )}{\partial q}$ = partial derivative of ( ) with respect to q
= differentiation with respect to span coordinate r
= differentiation with respect to time
[] = square matrix
{} = column matrix

Received April 11, 1973. Acknowledgment is made of the support of the U.S. Army Research Office, Durham, N.C., under Grant DA-AROD-31-12471G112

Index categories: Aircraft Vibration; VTOL Aircraft Design; VTOL Vibration.

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Introduction

THE modes of bending vibration of rotor, prop-rotor, and propeller blades would take place in the plane of steady rotation and normal to it, if the blades were untwisted and operated at flat pitch. However the aerodynamic inflow necessitates both twist and sometimes large pitch angles, especially in the cases of the propeller and prop-rotor blades. This results in strong elastic coupling of these modes by the inclination of the blade principal planes of bending along its span. Formidable numerical complications in the vibration analysis usually result. On the other hand, a suitable energy formulation of the problem is shown to provide a simple calculation procedure employing a correction function for modifying the idealized uncoupled modes of an untwisted blade rotating at flat pitch. This approach is especially well suited for preliminary design in that it permits a rapid evaluation of the effects of the aerodynamic requirements on the structural dynamic design and behavior of these blades.

Analysis

The Lagrangian method is employed and the analysis begins with an accounting of the kinetic and potential energies of the system. The latter is expressed in two parts: first, the strain energy in bending with respect to the two orthogonal but nonprincipal direction axes w and v; second, the potential energy stored by virtue of the centrifugal force field. The kinetic energy of the small oscillations with respect to the steadily rotating blade is given by

$$T = \frac{1}{2}\int_0^R m(\dot w^2 + \dot v^2)dr \tag{1}$$

The potential energy of the centrifugal force field is given by[2,3,7]

$$U_F = \frac{1}{2}\Omega^2\int_0^R [\sigma(w'^2 + v'^2) - mv^2]dr \tag{2}$$

$$\sigma = \int_r^R m\eta d\eta \tag{3}$$

The potential energy due to elastic bending about the nonprincipal elastic directions is

$$U_S = \frac{1}{2}\int_0^R \begin{pmatrix} [(EI)_{\bar w}\cos^2\theta + (EI)_{\bar v}\sin^2\theta]w'^2 \\ + [(EI)_{\bar w}\sin^2 + (EI)_{\bar v}\cos^2\theta]v'^2 \\ + [EI)_{\bar v} - (EI)_{\bar w}]\sin 2\theta w''v'' \end{pmatrix} dr \tag{4}$$

Expressing the vibratory displacements as

$$w(r, t) = q_w(t)W(r) \tag{5}$$

$$v(r, t) = q_v(t)V(r) \tag{6}$$

and the equations of motion in terms of the generalized coordinates[4] q_w and q_v, the Lagrangian form of the equations of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_w}\right) + \frac{\partial U_F}{\partial q_w} + \frac{\partial U_S}{\partial q_w} = 0 \tag{7}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_v}\right) + \frac{\partial U_F}{\partial q_v} + \frac{\partial U_S}{\partial q_v} = 0 \tag{8}$$

Writing these equations explicitly in a matrix format yields

$$\begin{bmatrix} m_w & 0 \\ 0 & m_v \end{bmatrix} \begin{Bmatrix} \ddot q_w \\ \ddot q_v \end{Bmatrix} + \begin{bmatrix} k_{ww} & k_{wv} \\ k_{vw} & k_{vv} \end{bmatrix} \begin{Bmatrix} q_w \\ q_v \end{Bmatrix} = 0 \tag{9}$$

The elastic coupling of the oscillations is evident from the matrix elements which follow.

$$m_w = \int_0^R mW^2 dr \quad (10)$$

$$m_v = \int_0^R mV^2 dr \quad (11)$$

$$k_{ww} = \Omega^2 \int_0^R \sigma W'^2 dr + \int_0^R [(EI)_{\bar{w}} \cos^2\theta + (EI)_{\bar{v}} \sin^2\theta] W''^2 dr \quad (12)$$

$$k_{vv} = \Omega^2 \int_0^R (\sigma V'^2 - mV^2) dr + \int_0^R [(EI)_{\bar{w}} \sin^2\theta + (EI)_{\bar{v}} \cos^2\theta] V''^2 dr \quad (13)$$

$$k_{wv} = k_{vw} = \frac{1}{2} \int_0^R [(EI)_{\bar{v}} - (EI)_{\bar{w}}] \sin 2\theta \, W'' V'' dr \quad (14)$$

When the blade is untwisted and at flat geometric pitch, the equations are uncoupled and we have the more familiar Rayleigh quotients[5] for the uncoupled natural frequencies

$$\omega_w^2 = \frac{\int_0^R (EI)_{\bar{w}} W''^2 dr + \Omega^2 \int_0^R \sigma W'^2 dr}{\int_0^R mW^2 dr} \quad (15)$$

$$\omega_v^2 = \frac{\int_0^R (EI)_{\bar{v}} V''^2 dr + \Omega^2 \int_0^R (\sigma V'^2 - mV^2) dr}{\int_0^R mV^2 dr} \quad (16)$$

We now make an approximation wherein the spanwise deflection functions $W(r)$ and $V(r)$ for the untwisted blades at flat pitch which satisfy Eqs. (15) and (16) are substituted in the general equations (9). They may then be written in the revealing form

$$\begin{bmatrix} \omega_w^2 + \Delta_w^2 - \omega^2, & \Delta_{wv}^2 \\ \Delta_{vw}^2, & \omega_v^2 - \Delta_v^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} \bar{q}_w \\ \bar{q}_v \end{Bmatrix} = 0 \quad (17)$$

where the natural frequencies ω are for the coupled out-of-plane and in-plane motions

$$q_w(t) = \bar{q}_w \sin\omega t \quad (18)$$

$$q_v(t) = \bar{q}_v \sin\omega t \quad (19)$$

and the new coefficients in the characteristic matrix are

$$\Delta_w^2 = \frac{\int_0^R [(EI)_{\bar{v}} - (EI)_{\bar{w}}] \sin^2\theta W''^2 dr}{m_v} \quad (20)$$

$$\Delta_v^2 = \frac{\int_0^R [(EI)_{\bar{v}} - (EI)_{\bar{w}}] \sin^2\theta V''^2 dr}{m_v} \quad (21)$$

$$\Delta_{wv}^2 = k_{wv}/m_w \quad (22)$$

$$\Delta_{vw}^2 = k_{vw}/m_v \quad (23)$$

The coupled natural frequencies follow at once from the roots of the biquadratic characteristic equation

$$\omega^4 - [(\omega_w^2 + \omega_v^2) + (\Delta_{\bar{w}}^2 \Delta_v^2)]\omega^2 + [\omega_w^2\omega_v^2 + \Delta_w^2\omega_v^2 - \Delta_v^2\omega_w^2 - \Delta_w^2\Delta_v^2 - \Delta_{wv}^2\Delta_{vw}^2] = 0 \quad (24)$$

In practice the deflection functions W and V will differ little and a useful approximation for the coupled natural frequencies follows as

$$\omega^4 - (\omega_w^2 + \omega_v^2)\omega^2 + [\omega_w^2\omega_v^2 - (\omega_w^2 - \omega_v^2)\Delta_w^2 - (\Delta_w^4 + \Delta_{wv}^4)] = 0 \quad (25)$$

Sample Calculation

Consider an untwisted prop-rotor blade designed to have fundamental out-of-plane and in-plane bending natural frequency ratios at flat pitch of 1.12 cycles per revolution, or $(\omega/\Omega)_w = (\omega/\Omega)_v = 1.12$ where the parameter ratios $[(EI)_w/m\Omega^2R^4] = 5 \times 10^{-3}$ and $[(EI)_v/m\Omega^2R^4] = 85 \times 10^{-3}$. Approximating the deflection functions by

$$W(r/R) \cong V(r/R) \cong 2(r/R)^2 - 4/3(r/R)^3 + 1/3(r/R)^4 \quad (26)$$

and considering a mean pitch angle of 45°, the out-of-plane frequency increases to 1.27 cycles per revolution and the in-plane frequency decreases to 0.79 cycles per revolution. In the limiting condition of a 90° inclination, the principal flexural rigidities are in effect interchanged and the high frequency, out-of-plane mode increases still further to 1.50 cycles per revolution, while the low frequency, in-plane mode is decreased in frequency to 0.49 cycles per revolution.

Conclusions

The influence of geometric twist and pitch in the bending vibrations of rotating beam-like structures such as hinge less rotor, prop-rotor, and propeller blades can be easily evaluated and undesirable combinations of flexural rigidity and other structural dynamic parameters can be avoided, if such calculations are made in the preliminary design stage. It is also evident from the sample calculation that a system which is acceptable for operation at low pitch settings such as a prop-rotor in its hovering flight condition may pass into an undesirable or even dangerous operating condition as the pitch is increased to the large angles consistent with its aerodynamic inflow in cruising or high speed propeller flight conditions. In particular the reduction in the in-plane natural frequency to one cycle per revolution and still lower frequencies as pitch increases may result either in a serious forced vibration due to resonance with once per revolution cyclic aerodynamic loading (arising from a slightly yawed inflow), or in dynamically unstable couplings with nacelle-wing-body oscillations.[6,7] A less powerful but sometimes significant coupling effect also present in hingeless rotor and prop-rotor blades is the Coriolis acceleration due to built-in coning of such blades.[7] This coupling effect due to coning and the geometric twist and pitch required for the large inflow conditions of the propeller state are now seen to be factors of importance in the design of advanced rotorcraft or convertible aircraft.

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